Graph Theory: Week 7

Directed graphs

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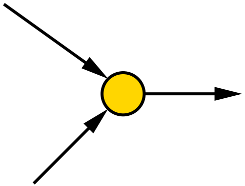
Directed graphs

- Also caled *digraphs*
- $D = (V, A, \psi)$
  where V are vertices, A are arcs, ψ is an incidence function giving an ordered pair of vertices for each arc.
- If $\psi(a) = (u, v)$, then arc a is a connection from u to v.
- Every digraph D has an underlying undirected graph G.
- We say that ψ is an *orientation* of G.
- Exercise: how many orientations does a graph G have?

Incidence

- The notion of “degree” needs to be generalised in a digraph. We talk about indegree and outdegree: $d^-(v)$, and $d^+(v)$.



$$d^-(v) = 2$$
$$d^+(v) = 1$$

- The maximum indegree and outdegree are denoted by Δ^- and Δ^+, while the minima are denoted by δ^- and δ^+.
- Exercise: show that $\sum_{v \in V} d^-(v) = \varepsilon = \sum_{v \in V} d^+(v)$.

Tournaments

- A tournament is an orientation of a complete graph.
- Imagine each vertex is a football team, and the direction of the arc specifies who won the match.
- Tournaments have some interesting properties…
- If *D* is a tournament then $\chi = \nu$ (a colouring requires as many colours as there are vertices).
- A tournament always has at least one vertex *v* from which there is a directed path of length ≤ 2 to every other vertex (proof by induction).

Euler tours in digraphs

- Previously we established that for there to be an Euler tour in an undirected graph G, each vertex must have even degree.
- What do you think the equivalent condition is for a digraph to be Eulerian?

Application: designing a rotational position sensor

- A large rotating cylinder is to be made as part of an industrial process. We need to know the position that the drum is in by using a sensor on the side of the cylinder.

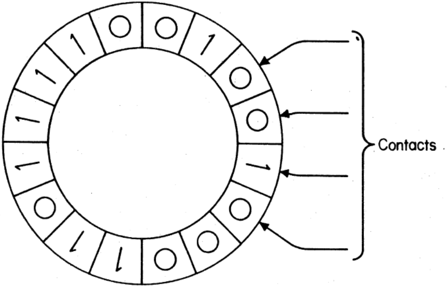


Figure 10.9. A computer drum

(Fig from Bondy & Murty)

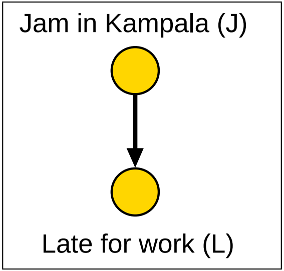
- Copper patches are used to make connections with contacts (1's). How should they be arranged?

Calculating switch positions

- Represent each combination of switch settings as a node in a digraph.
- For n sensors, we have a binary number with $n-1$ digits: 2^{n-1} combinations.
- There is an arc from node $p_1 p_2, \ldots p_{n-1}$ to node $p_2 p_3 \ldots p_n$ for all p_1, p_n.
  Give each such arc the label $p_1 p_2 p_3 \ldots p_n$.
  (example on board for $k = 2$).
- Each vertex v has $d^-(v) = d^+(v) = 2$. Therefore there is an Euler tour. Finding it gives a solution (simply read off the arc labels on the tour).
- Exercise: work out the solution for $k = 3$.

A different application of digraphs: representing probability distributions

- Each vertex represents a variable which can have different settings. *J* and *L* can both be either true or false.



Probability of a jam

P(J=True)	0.4
P(J=False)	0.6

Probability of being late

P(L=True|J=True)	0.9
P(L=False|J=True)	0.1
P(L=True|J=False)	0.2
P(L=False|J=False)	0.8

- What is the probability of being late for work if you don't know whether there's a jam or not?

Probability of being late

$$\mathbf{P}(\mathbf{L} = \mathbf{True})$$

$$
\begin{aligned}
&= P(L{=}True|J{=}True)P(J{=}True) + P(L{=}True|J{=}False)P(J{=}False) \\
&= 0.9 \times 0.4 + 0.2 \times 0.6 \\
&= 0.36 + 0.12 \\
&= 0.48
\end{aligned}
$$

- Therefore people would be expected to be late about half the time.

Inference in a graphical model

- Say your colleague is late for work. What is the probability that there is a jam?
- We are now trying to find $P(J = \mathit{True} | L = \mathit{True})$.
- This involves the use of Bayes' rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

- Find the required probability using the tables on the previous slide.

Probability of a jam given lateness

$$\mathbf{P}(\mathbf{J} = \mathbf{True}|\mathbf{L} = \mathbf{True})$$

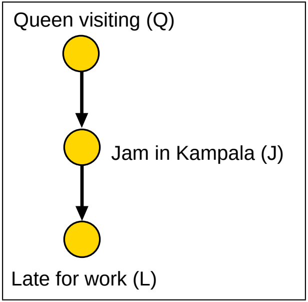
$$= \frac{P(L{=}True|J{=}true)P(J{=}true)}{P(L{=}True|J{=}true)P(J{=}true)+P(L{=}True|J{=}False)P(J{=}False)}$$

$$= \frac{0.9 \times 0.4}{0.9 \times 0.4 + 0.2 \times 0.6}$$

$$= \frac{0.36}{0.36 + 0.12}$$

$$= 0.75$$

Adding vertices to the graphical model



- How would you calculate the probability of being late now?
- How would you calculate the probability that the queen is visiting if you notice that one of your colleagues is late?
- Can keep adding vertices to the digraph. The direction of the arcs indicates conditional dependencies on the values of variables.
- If the values of some variables are known, repeated application of Bayes' rule can be used to make inferences.