Graph Theory: Week 5

Matchings and coverings

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Recap: Euler tours

- What are the conditions for a graph to contain an Euler tour?
- What's the difference between an Euler cycle/tour and a Hamilton cycle?

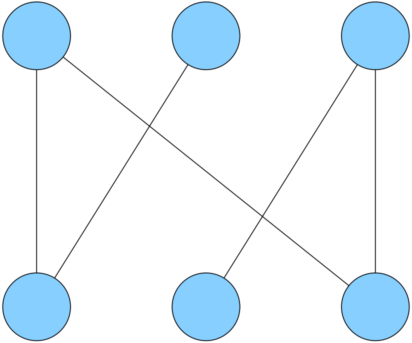
Matching problems

- Some real-world problems involve finding matching pairs in a group.
- For example, we might want to allocate jobs to candidates. There are a number of candidates who are qualified for each job; what is the arrangement which leaves all positions filled?
- What if the candidates are qualified for different jobs to different extents—i.e. some matchings are preferable to others?

Matchings

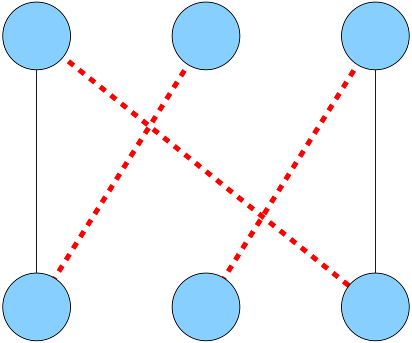
- A *matching* is a subset of edges in a graph which have no common vertices.
- For each edge M in a matching, the two vertices at either end are matched.
- A *maximum matching* is one in which as many vertices are matched as possible.
- A *perfect matching* is one in which every vertex is matched.
- An *M-alternating* path in a graph is one in which the edges are alternately in M and $G \backslash M$.

Example: this bipartite graph…

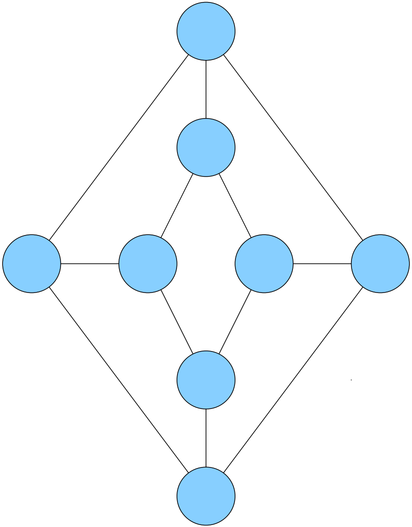


. . . has this (perfect) matching.

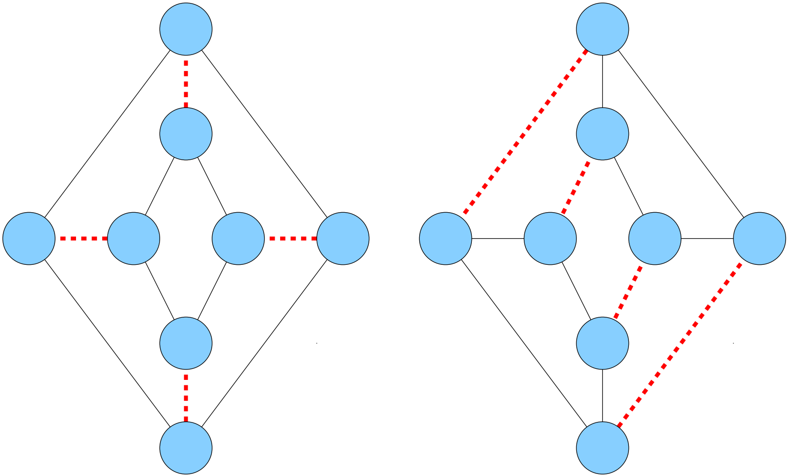
Matching $M \subseteq E$ shown as dashed lines:



Does this graph have a matching?



Yes



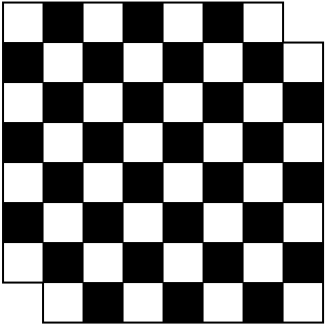
- In fact, this graph has several matchings.

Perfect matches in trees

- A tree has a maximum of one perfect match.
- Prove this.

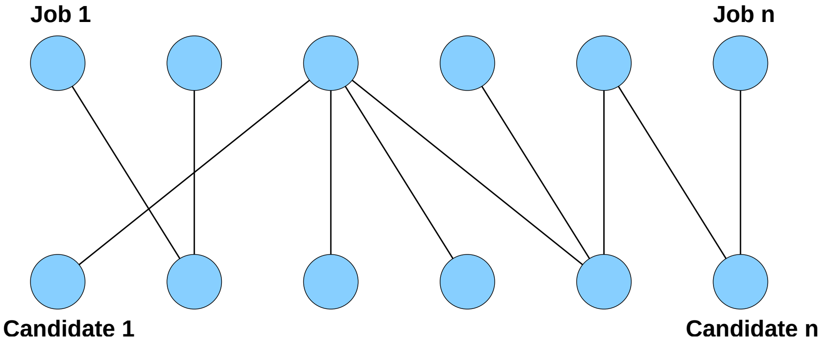
Matches in bipartite graphs

- An 8×8 square grid has the corner tiles removed.
- Is it possible to cover the whole of the remaining grid with 2×1 blocks?



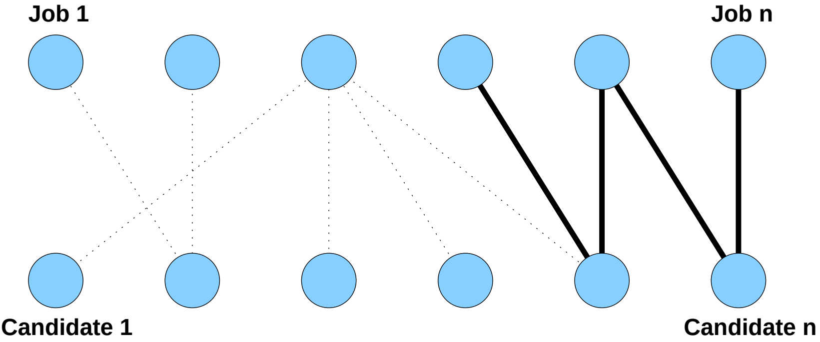
Personnel assignment

- There are a group of jobs (J) and a group of candidates (C).
- Each candidate is qualified for one or more of the jobs.
- Is it possible to find a perfect matching?



Hall's theorem

- For a subset $S \subseteq V$ of vertices in a graph, define the *neighbourhood* as all the vertices which are adjacent to S.
- There is a matching in the graph if for all subsets S, $|N(S)| \geq |S|$.



- Here there is a subset of jobs for which |N(S)| < |S|.

The marriage theorem

- In a k-regular bipartite graph with k non-zero, there is a perfect matching.
- So, if there are a group of men M, and a group of women W, and every man knows k women, then everybody can be paired off with someone they know.
- Proof:
  - If all the vertices in M and W have the same degree, then $|M| = |W|$.
  - Every subset $S \subseteq M$ has $k|S|$ edges incident to it.
  - Therefore the subset S has a neighbourhood containing at least $|S|$ vertices.
  - Therefore $|N(S)| \geq |S|$.
  - Therefore there is a perfect matching according to Hall's theorem.

The stable marriage problem

- Let's say a matching has been made which includes $\{M_1, W_1\}$ and $\{M_2, W_2\}$.
- What if $\{M_1, W_2\}$ would prefer each other and so would $\{M_2, W_1\}$? Trouble
- The Gale-Shapley algorithm solves this problem, so that matchings are found in which there are no unstable pairs of couples.

The Gale-Shapley algorithm

While there is still a man *m* who has not become engaged and still has somebody to propose to:

1. Choose *w*, the woman he would most prefer who he has not yet proposed to.
2. If *w* is free
   - *m* and *w* become engaged.
3. Else *w* is already paired with m^*
   - If *w* prefers *m* over her existing choice
     - *m* and *w* become engaged.
     - m^* becomes free.
   - Otherwise, there is no change.

Stable marriages

- Does the preceeding algorithm favour men or women?
- Applications?
- Ugandan stable marriage problem?

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