Graph Theory: Week 2

Trees

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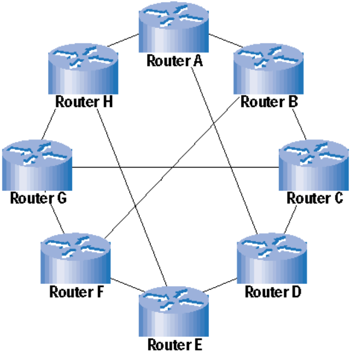
Connecting towns

- If you were building a rail network to connect all the towns in Uganda, which arrangement would involve the least amount of track to be laid?

Connecting nodes on a network

- How many different topologies are possible to connect a number of routers on a network, and which are preferable?



- Once again, graph theory makes difficult problems like these easier.

Refs for this week

- This week's material loosely follows chapter 2 of Bondy and Murty, "Graph Theory with Applications".
- Local copies will be kept on `blackboard.mak.ac.ug`.

Trees

Trees are graphs which are:

- Acyclic
- Connected
- Simple

Edges and vertices in a tree

- Every graph has a number of edges, ε.
- Every graph also has a number of vertices, ν.
- Exercise: prove that for a tree,

$$\varepsilon = \nu - 1$$

- Hint: think about the simplest tree first, then add vertices one by one (proof by induction).

Edges and vertices in a tree (cont.)

- Therefore, if any simple, connected graph has $\nu - 1$ edges, it must be a tree.
- Can you see why this is true?

The degree of a vertex

- The degree of a vertex $d(v)$ is the number of edges which are incident to it.
- Exercise: show that

$$\sum_{v \in V} d(v) = 2\varepsilon$$

Degrees of vertices in a tree

Given the previous result, show that there are at least two vertices in any tree with degree 1.
(Bondy and Murty, p26).

Cut edges

- A **cut edge** is an edge which would disconnect a graph if it was removed.
- How many cut edges are there in a tree?

Minimum spanning trees

- Sometimes we are interested in finding a **subgraph** which connects every vertex in the graph using the lowest number of edges.
- **Kruskal's algorithm** provides a way of doing this.
- Example, connecting electricity lines between houses.

Spanning tree exercise

Bondy and Murty, p37.

Proof that Kruskal's algorithm works

How can you be sure that this algorithm always gives an optimal spanning tree?

One more question about spanning trees

What might you do if you wanted to consider how popular the route between each pair of cities was, in addition to how easy it would be to connect them?

When are two graphs equivalent?

- If two graphs have the same structure, but with different labels, or drawn in a different way, they are **isomorphic**.
- Sometimes it is easy to tell if two graphs are not isomorphic (e.g. because they have a different number of vertices)...
- ...but sometimes it is difficult.

Subgraph exercise

How many non-isomorphic spanning trees are there in the example?
(Bondy and Murty, exercise 2.2.7).